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ORIGINAL PAPER

A mathematical characterization of the groups of substitution isomerism of the linear alkanes

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Abstract In this paper, a mathematical characterization of Lunn–Senior's groups of univalent substitution isomerism of the linear alkanes, under some natural assumptions that reflect their common properties, is given. For each linear alkane, the number of its monosubstitution derivatives, its di-substitution derivatives, and its tri-substitution derivatives as linear, quadratic, and cubic polynomial expressions, respectively, in their number, is obtained. In principle, the number of derivatives of a given linear alkane with any particular composition can be established. The same explicit expressions for the case of k-substitution homogeneous derivatives of the linear alkanes are obtained by Balasubramanian (Theoret. Chim. Acta (Berl.) 51:37, 1979).

Keywords Linear alkanes · Lunn–Senior's group of substitution isomerism · Number of substitution derivatives

The main result of the present paper is Theorem 2 which characterizes Lunn–Senior's groups of univalent substitution isomerism of the linear alkanes up to conjugation in the corresponding symmetric group. This is done if one postulates four natural properties of these groups that are reflections of the common chemical properties of the corresponding compounds.

Theorem 2 and [7, Corollary 5.2.9] yield two corollaries, which establish the numbers of all monosubstitution, di-substitution and tri-substitution derivatives of n-th linear alkane, $n \geq 2$, as polynomials in n, as well as some linear relations among these polynomials. Using the general formula from [7, Corollary 5.2.9], one can find, in principle, the number of derivatives of n-th linear alkane with any particular composition

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as a polynomial expression in n. In the case of k-substitution homogeneous derivatives of n-th linear alkane $C_{2n+2-k}X_k$, these expressions can be obtained also from formulae (9) and (10) in [1]. There K. Balasubramanian uses Pólya's theorem, applied for the group G_n derived as point group of symmetries of the molecule of n-th linear alkane. The formula from [7, Corollary 5.2.9] gives an explicit expression of the coefficients of the cycle index of the group G_n before the monomial symmetric functions. The paper [3] contains a computer program for finding these coefficients. For examples of application of Pólya's theorem and its generalizations in a broader context the reader can see the remarkable review paper [4] and its references, from which we note the important book [5]. The case of poly-substituted alcohols is considered in [2].

As a rule, the first member of a homologous series has properties that distinguish it from all other members of this series that have common properties. The homologous series of linear alkanes is not an exception—for instance, the C_1-alkane, methane, has the alternating group A_4 of order 12 as its group of univalent substitution isomerism, whereas in accord with [9, IV], or [8, Theorem 1.4.1], the corresponding group

$$G_2 = \langle (123), (456), (14)(25)(36) \rangle \tag{1}$$

of the C_2-alkane, ethane, has order 18. In order to reflect the common properties of the linear C_n-alkanes C_nH_{2n+2}, $n \geq 2$, Lunn and Senior supposed in [9, IV] that their groups $G_n \leq S_{2n+2}$ of univalent substitution isomerism have the form

$$G_n = \langle (123), (456), (14)(25)(36)(78)\ldots(2n+1, 2n+2) \rangle$$

Thus, $G_2 \simeq G_3 \simeq \cdots \simeq G_n \simeq \cdots$, and all these groups have order 18. Moreover,

$$G_n \leq S_{[1,6]} \times S_{[7,8]} \times \cdots \times S_{[2n+1,2n+2]},$$

the projection

$$S_{[1,6]} \times S_{[7,8]} \times \cdots \times S_{[2n+1,2n+2]} \to S_{[1,6]}$$

induces an isomorphism $G_n \to G_2$, and the projection

$$S_{[1,6]} \times S_{[7,8]} \times \cdots \times S_{[2n+1,2n+2]} \to S_{[7,8]} \times \cdots \times S_{[2n+1,2n+2]}$$

induces a surjective homomorphism $G_n \to H_2 = \langle (78)\ldots(2n+1, 2n+2) \rangle$ with kernel $H = \langle (123), (456) \rangle$. Since $G_n \leq G_2 \times H_2$, the group G_n is a sub-direct product of the groups G_2 and H_2, and $G_n/H \simeq H_2$, [7, 5.5]. In particular, the group G_n is $(n-1)$-transitive, that is, we have $\nu_{(2n+1,1);G_n} = n - 1$.

Here, we make weaker assumptions about the groups $G_n \leq S_{2n+2}$ of univalent substitution isomerism of the linear $C_n$alkanes, $n \geq 2$, which mirror back both their differences, as well as their common properties:

(i_n) the group G_n is $(n-1)$-transitive;
(ii_n) the fixed-point set of G_n is empty;
(iii_n) there exists a set of transitivity $X \subset [1, 2n+2]$ of G_n, which contains at least 6 elements;
(iv_n) the group G_n has order 18, and the restriction homomorphism $S_{2n+2} \to S_X$ induces an injective homomorphism $G_n \to S_X$.

Condition (i_n) means that when the formula of these alkanes increases uniformly by a CH_2 increment, then the number of sets of transitivity increases by 1, thus underlining the differences. Condition (ii_n) means that no privilege is granted to any single free valence. On the other hand, properties (iii_n) and (iv_n) of the group G_n reflect the similarities of the linear alkanes with ethane.

These conditions are enough for the group G_n to be characterized up to conjugation in the symmetric group S_{2n+2}.

Theorem 2 *The group $G_n \leq S_{2n+2}$ of univalent substitution isomerism of the linear n-th alkane, $n \geq 2$, coincides up to conjugation with the group*

$$\langle(123), (456), (14)(25)(36)(78)\dots(2n+1, 2n+2)\rangle. \tag{3}$$

Conversely, the group (3) satisfies conditions (i_n–iv_n).

Corollary 4 *For any $n \geq 2$ one has the following numbers of products of the n-th linear alkane:*

(*i*) *the number of monosubstitution products is* $\nu_{(2n+1,1);G_n} = n-1$;
(*ii*) *the number of di-substitution homogeneous products is* $\nu_{(2n,2);G_n} = n^2-2n+2$;
(*iii*) *the number of di-substitution heterogeneous products is*

$$\nu_{(2n,1^2);G_n} = 2n^2 - 5n + 5;$$

(*iv*) *the number of tri-substitution homogeneous products is*

$$\nu_{(2n-1,3);G_n} = \frac{1}{3}(2n^3 - 9n^2 + 19n - 12);$$

(*v*) *the number of tri-substitution products with composition $C_nH_{2n-1}X_2Y$ is*

$$\nu_{(2n-1,2,1);G_n} = 2n^3 - 9n^2 + 19n - 14;$$

(*vi*) *(vi) the number of tri-substitution products with composition $C_nH_{2n-1}XYZ$ is*

$$\nu_{(2n-1,1^3);G_n} = 4n^3 - 18n^2 + 38n - 28.$$

Corollary 5 *One has the following linear relations, where n is any integer ≥ 2:*

(*i*) $\nu_{(2n,1^2);G_n} - 2\nu_{(2n,2);G_n} + \nu_{(2n+1,1);G_n} = 0$;
(*ii*) $3\nu_{(2n-1,3);G_n} - \nu_{(2n-1,2,1);G_n} = 2$;

(iii) $\nu_{(2n-1,1^3);G_n} = 2\nu_{(2n-1,2,1);G_n}$.

Remark 6 The table below contains the numbers of all monosubstitution, di-substitution, and tri-substitution derivatives of $n -$ th linear alkane for $2 \leq n \leq 10$.

	$(2n+1,1)$	$(2n,2)$	$(2n,1^2)$	$(2n-1,3)$	$(2n-1,2,1)$	$(2n-1,1^3)$
$n=2$ (ethane)	1	2	3	2	4	8
$n=3$ (propane)	2	5	8	6	16	32
$n=4$ (butane)	3	10	17	16	46	92
$n=5$ (pentane)	4	17	30	36	106	212
$n=6$ (hexane)	5	26	47	70	208	416
$n=7$ (heptane)	6	37	68	122	364	728
$n=8$ (octane)	7	50	93	196	586	1172
$n=9$ (nonane)	8	65	122	296	886	1772
$n=10$ (decane)	9	82	155	426	1276	2552

In order to prove Theorem 2 we need some preliminary results. Any permutation group $G \leq S_d$ produces a tabloid $D = (D_1, D_2, \ldots)$ by ordering the sets of transitivity $D_1, D_2, \ldots$ of G from largest to smallest, and the tabloid D produces, in turn, a partition $\delta = (\delta_1, \delta_2, \ldots)$ of d, where $\delta_k = |D_k|$. The length $\ell = \ell(\delta)$ of the partition δ is equal to the number of sets of transitivity of G. If we write δ in the form $\delta = (1^{m_1}, 2^{m_2}, \ldots, d^{m_d})$, then we obtain the linear Diofantine system

$$\begin{aligned} m_1 + m_2 + \cdots + m_d &= \ell \\ m_1 + 2m_2 + \cdots + dm_d &= d \end{aligned} \tag{7}$$

in the unknowns $m_1, m_2, \ldots, m_d$.

The lemma below characterizes the possible partitions δ under certain assumptions.

Lemma 8 *If the permutation group $G \leq S_d$ has ℓ sets of transitivity and its fixed-point set is empty, and if $d = 2\ell + 4$, then the partition δ of d is equal to one of the following partitions*:

$$(2^{\ell-1}, 6^1), \quad (2^{\ell-2}, 3^1, 5^1), \quad (2^{\ell-2}, 4^2), \quad (2^{\ell-3}, 3^2, 4^1), \quad (2^{\ell-4}, 3^4).$$

Proof The condition yields $m_1 = 0$, and let $k \geq 2$ be the largest index, such that $m_k \geq 1$. Now, the system (7) becomes

$$\begin{aligned} m_2 + \cdots + m_k &= \ell \\ 2m_2 + \cdots + km_k &= 2\ell + 4 \end{aligned} \tag{9}$$

The system

$$\begin{aligned} m_2 + m_3 + \cdots + m_k &= \ell \\ m_3 + \cdots + (k-2)m_k &= 4 \end{aligned}$$

is equivalent to (9), and implies $k \leq 6$.

If $k = 6$, then the system (9) has a unique solution

$$(m_2, m_3, m_4, m_5, m_6) = (\ell - 1, 0, 0, 0, 1).$$

If $k = 5$, then (9) has a unique solution

$$(m_2, m_3, m_4, m_5, m_6) = (\ell - 2, 1, 0, 1, 0).$$

In case $k = 4$, this system has two solutions

$$(m_2, m_3, m_4, m_5, m_6) = (\ell - 2, 0, 2, 0, 0),\ (\ell - 3, 2, 1, 0, 0).$$

If $k = 3$, then we obtain

$$(m_2, m_3, m_4, m_5, m_6) = (\ell - 4, 4, 0, 0, 0).$$

In case $k = 2$ the system (9) is inconsistent and we finish the proof. □

Lemma 10 *If $G \leq S_6$ is a transitive permutation group of order 18, then G is conjugated to the group G_2 from (1).*

Proof Let $H \leq G$ be a Sylow 3-subgroup of G. According to Sylow theorems [6, 4.2] the number of Sylow 3-subgroup of G is a divisor of 18 of the type $3k + 1$, so H is a normal subgroup of G. Since $|H| = 3^2$, [6, 4.4] yields that the group H is elementary Abelian of type $(3, 3)$. After eventual conjugation in S_6, we can suppose $H = \langle(123), (456)\rangle$. Sylow theorems also yield that there are Sylow 2-subgroup of G, and let $\langle\iota\rangle$ is one of them. The generator $\iota \in G$ has order 2, and $G = H\langle\iota\rangle$. We have either $\iota\langle(123)\rangle\iota = \langle(123)\rangle$, or $\iota\langle(123)\rangle\iota = \langle(456)\rangle$. In the first case $\iota\langle(456)\rangle\iota = \langle(456)\rangle$ and the sets $B_1 = \{1, 2, 3\}$ and $B_2 = \{4, 5, 6\}$ are two sets of transitivity of G—a contradiction. In the second case $\iota\langle(456)\rangle\iota = \langle(123)\rangle$, and, in particular, we obtain $\iota B_1 = B_2$. Therefore $\iota \in \Omega$, where

$$\Omega = \{(14)(25)(36), (15)(26)(34), (16)(24)(35),\\ (14)(26)(35), (15)(24)(36), (16)(25)(34)\}.$$

The set Ω is stable with respect to conjugation with elements of the group H and there are two H-orbits in Ω:

$$\Omega' = \{(14)(25)(36), (15)(26)(34), (16)(24)(35)\},$$

and

$$\Omega'' = \{(14)(26)(35), (15)(24)(36), (16)(25)(34)\}.$$

We have either $\iota \in \Omega'$, or $\iota \in \Omega''$, hence we can set $\iota = (14)(25)(36)$, or $\iota = (14)(26)(35)$. Thus, $G = H\langle(14)(25)(36)\rangle$, or $G = H\langle(14)(26)(35)\rangle$. Since the last two groups are conjugated in S_6 via the transposition (56), we are done. □

Proof of Theorem 2 Lemma 8 for $d = 2n+2$ and $\ell = n-1$ yields that any group $G \leq S_{2n+2}$ that satisfies the conditions (i_n–iv_n) is a subgroup of one of the following direct products: $S_6 \times (S_2)^{n-2}$, $S_5 \times S_3 \times (S_2)^{n-3}$, $S_4 \times S_4 \times (S_2)^{n-3}$, $S_4 \times S_3 \times S_3 \times (S_2)^{n-4}$, or $S_3 \times S_3 \times S_3 \times S_3 \times (S_2)^{n-5}$. Condition ($\text{iii}_n$) excludes all possibilities except the first one. After an eventual conjugation in S_{2n+2}, we can assume that

$$G \leq S_{[1,6]} \times S_{[7,8]} \times \cdots \times S_{[2n+1,2n+2]},$$

and, in accord with (iv_n), that the projection

$$S_{[1,6]} \times S_{[7,8]} \times \cdots \times S_{[2n+1,2n+2]} \to S_{[1,6]}$$

induces an isomorphism $\pi_1 : G \to \tilde{G}_2$ where $\tilde{G}_2$ is the transitive image of G. Lemma 10 implies that $\tilde{G}_2$ is conjugated in $S_6 = S_{[1,6]}$ to the group G_2 from (1), and hence, after eventual conjugation in S_{2n+2} with a permutation from S_6, we can suppose that $\tilde{G}_2 = G_2$. Let us consider the image Λ_2 of the group G via the projection

$$S_{[1,6]} \times S_{[7,8]} \times \cdots \times S_{[2n+1,2n+2]} \to S_{[7,8]} \times \cdots \times S_{[2n+1,2n+2]}$$

which induces a surjective homomorphism $\pi_2 : G \to \Lambda_2$. The integer-valued intervals $[7, 8], \ldots, [2n+1, 2n+2]$, are the remaining $n-2$ sets of transitivity of G, and hence, of Λ_2. In particular, Λ_2 is non-trivial and its order is a power of 2 that divides $|G| = 18$. Therefore $|\Lambda_2| = 2$, and $\Lambda_2 = \langle (78) \ldots (2n+1, 2n+2) \rangle$. The kernel H of the projection π_2 is the only Sylow $3-$subgroup of G, and because of the isomorphism $\pi_1 : G \to G_2$ we have $H = \langle (123), (456) \rangle$. All permutations in the difference $G \backslash H$ contain the transpositions $(78), \ldots, (2n+1, 2n+2)$, and, moreover, $\pi_1(G \backslash H) = H(14)(25)(36)$. Therefore $G \backslash H = H(14)(25)(36)(78) \ldots (2n+1, 2n+2)$, so the group G coincides with (3). A straightforward check shows that the group (3) satisfies (i_n) – (iv_n). □

Proof of Corollary 4 We will use the notation from [7, 5.2] as well as [7, Corollary 5.2.9].

(i) It is enough to note that $\nu_{(2n+1,1);G_n}$ is the number of sets of transitivity of G_n.
(ii) First, let us suppose $n \geq 3$. Then

$$\begin{aligned} L\,(G_n, S_{2n} \times S_2) = \Big\{ &\Big(\big(2^{n+1}\big), \big(2^n\big), \big(2^1\big)\Big), \Big(\big(3^1, 1^{2n-1}\big), \\ &\big(3^1, 1^{2n-3}\big), \big(1^2\big)\Big), \Big(\big(3^2, 1^{2n-4}\big), \big(3^2, 1^{2n-6}\big), \big(1^2\big)\Big), \\ &\Big(\big(6^1, 2^{n-2}\big), \big(6^1, 2^{n-3}\big), \big(2^1\big)\Big) \Big\}, \end{aligned}$$

and hence

$$\begin{aligned}\nu_{(2n,2);G_n} &= \frac{1}{18}\left(\frac{(2n+2)!}{(2n)!2!}+3\frac{z_{(2^{n+1})}}{z_{(2^n)}z_{(2^1)}}+4\frac{z_{(3^1,1^{2n-1})}}{z_{(3^1,1^{2n-3})}z_{(1^2)}}\right.\\ &\quad\left.+4\frac{z_{(3^2,1^{2n-4})}}{z_{(3^2,1^{2n-6})}z_{(1^2)}}+6\frac{z_{(6^1,2^{n-2})}}{z_{(6^1,2^{n-3})}z_{(2^1)}}\right)\\ &= \frac{1}{18}\left(\frac{(2n+2)(2n+1)}{2}+3\frac{2^{n+1}(n+1)!}{2^n n!2^1 1!}\right.\\ &\quad+4\frac{3^1 1!1^{2n-1}(2n-1)!}{3^1 1!1^{2n-3}(2n-3)!1^2 2!}+4\frac{3^2 2!1^{2n-4}(2n-4)!}{3^2 2!1^{2n-6}(2n-6)!1^2 2!}\\ &\quad\left.+6\frac{6^1 1!2^{n-2}(n-2)!}{6^1 1!2^{n-3}(n-3)!2^1 1!}\right)\\ &= \frac{1}{18}\left((n+1)(2n+1)+3(n+1)+4(n-1)(2n-1)\right.\\ &\quad\left.+4(n-2)(2n-5)+6(n-2)\right)=n^2-2n+2.\end{aligned}$$

If $n=2$, then

$$L(G_2, S_4\times S_2)=\{((2^3),(2^2),(2^1)),((3^1,1^3),(3^1,1^1),(1^2))\},$$

so we get $\nu_{(4,2);G_2}=2$, and this the value of the polynomial n^2-2n+2 for $n=2$.

(iii) Let us suppose that $n\geq 3$. We have

$$\begin{aligned}L\left(G_n, S_{2n}\times S_1\times S_1\right) = \Big\{&\left(\left(3^1,1^{2n-1}\right),\left(3^1,1^{2n-3}\right),\left(1^1\right),\left(1^1\right)\right),\\ &\left(\left(3^2,1^{2n-4}\right),\left(3^2,1^{2n-6}\right),\left(1^1\right),\left(1^1\right)\right)\Big\},\end{aligned}$$

and therefore

$$\begin{aligned}\nu_{(2n,1^2);G_n} &= \frac{1}{18}\left(\frac{(2n+2)!}{(2n)!1!1!}+4\frac{z_{(3^1,1^{2n-1})}}{z_{(3^1,1^{2n-3})}z_{(1^1)}z_{(1^1)}}+4\frac{z_{(3^2,1^{2n-4})}}{z_{(3^2,1^{2n-6})}z_{(1^1)}z_{(1^1)}}\right)\\ &= \frac{1}{18}\left((2n+2)(2n+1)+4\frac{3^1 1!1^{2n-1}(2n-1)!}{3^1 1!1^{2n-3}(2n-3)!1^1 1!1^1 1!}\right.\\ &\quad\left.+4\frac{3^2 2!1^{2n-4}(2n-4)!}{3^2 2!1^{2n-6}(2n-6)!1^1 1!1^1 1!}\right)\\ &= \frac{1}{18}\left((2n+2)(2n+1)+4(2n-1)(2n-2)+4(2n-4)(2n-5)\right)\\ &= n^2-5n+5.\end{aligned}$$

In case $n = 2$, we have

$$L(G_2, S_4 \times S_1 \times S_1) = \{((3^1, 1^1), (1^1), (1^1))\},$$

so we obtain $\nu_{(4,1^2);G_2} = 3$, and this also is the value of the polynomial $2n^2 - 5n + 5$ at $n = 2$.

(iv) Here we suppose $n \geq 4$. We have

$$\begin{aligned}&L(G_n, S_{2n-1} \times S_3) =\\ &\{((3^1, 1^{2n-1}), (3^1, 1^{2n-4}), (1^3)), ((3^1, 1^{2n-1}), (1^{2n-1}), (3^1)),\\ &((3^2, 1^{2n-4}), (3^1, 1^{2n-4}), (3^1)), ((3^2, 1^{2n-4}), (3^2, 1^{2n-7}), (1^3))\},\end{aligned}$$

and hence we get

$$\begin{aligned}\nu_{(2n-1,3);G_n} &= \frac{1}{18}\left(\frac{(2n+2)!}{(2n-1)!3!} + 4\frac{z_{(3^1,1^{2n-1})}}{z_{(3^1,1^{2n-4})}z_{(1^3)}} + 4\frac{z_{(3^1,1^{2n-1})}}{z_{(1^{2n-1})}z_{(3^1)}}\right.\\ &\quad\left. + 4\frac{z_{(3^2,1^{2n-4})}}{z_{(3^1,1^{2n-4})}z_{(3^1)}}\right) + 4\frac{z_{(3^2,1^{2n-4})}}{z_{(3^2,1^{2n-7})}z_{(1^3)}}\\ &= \frac{1}{18}\left(\frac{(2n+2)(2n+1)(2n)}{6} + 4\frac{3^1 1!1^{2n-1}(2n-1)!}{3^1 1!1^{2n-4}(2n-4)!1^3 3!}\right.\\ &\quad + 4\frac{3^1 1!1^{2n-1}(2n-1)!}{1^{2n-1}(2n-1)!3^1 1!}\\ &\quad\left. + 4\frac{3^2 2!1^{2n-4}(2n-4)!}{3^1 1!1^{2n-4}(2n-4)!3^1 1!} + 4\frac{3^2 2!1^{2n-4}(2n-4)!}{3^2 2!1^{2n-7}(2n-7)!1^3 3!}\right)\\ &= \frac{1}{18}\left(\frac{2}{3}(2n+1)(n+1)n + \frac{2}{3}(2n-1)(2n-2)(2n-3)\right.\\ &\quad\left. + \frac{2}{3}(2n-4)(2n-5)(2n-6) + 12\right)\\ &= \frac{1}{3}(2n^3 - 9n^2 + 19n - 12).\end{aligned}$$

Now, let $n = 3$ or $n = 2$. Then

$$\begin{aligned}L\left(G_n, S_{2n-1} \times S_3\right) = \Big\{&\Big(\Big(3^1, 1^{2n-1}\Big), \Big(3^1, 1^{2n-4}\Big), \Big(1^3\Big)\Big), \Big(\Big(3^1, 1^{2n-1}\Big),\\ &\Big(1^{2n-1}\Big), \Big(3^1\Big)\Big), \Big(\Big(3^2, 1^{2n-4}\Big), \Big(3^1, 1^{2n-4}\Big), \Big(3^1\Big)\Big)\Big\},\end{aligned}$$

and we obtain $\nu_{(5,3);G_3} = 6$, $\nu_{(3,3);G_2} = 2$, and these numbers are the values of the polynomial $\frac{1}{3}(2n^3 - 9n^2 + 19n - 12)$ at $n = 3$ and $n = 2$, respectively.

(v) First, let us suppose $n \geq 4$. We have

$$L(G_n, S_{2n-1} \times S_2 \times S_1) = \{((3^1, 1^{2n-1}), (3^1, 1^{2n-4}), (1^2), (1^1)), ((3^2, 1^{2n-4}), (3^2, 1^{2n-7}), (1^2), (1^1))\},$$

and hence we obtain

$$\begin{aligned}
\nu_{(2n-1,2,1);G_n} &= \frac{1}{18}\left(\frac{(2n+2)!}{(2n-1)!2!1!} + 4\frac{z_{(3^1,1^{2n-1})}}{z_{(3^1,1^{2n-4})}z_{(1^2)}z_{(1^1)}}\right.\\
&\quad \left. + 4\frac{z_{(3^2,1^{2n-4})}}{z_{(3^2,1^{2n-7})}z_{(1^2)}z_{(1^1)}}\right)\\
&= \frac{1}{18}\left(\frac{(2n+2)(2n+1)(2n)}{2} + 4\frac{3^1 1!1^{2n-1}(2n-1)!}{3^1 1!1^{2n-4}(2n-4)!1^2 2!1^1 1!}\right.\\
&\quad \left. + 4\frac{3^2 2!1^{2n-4}(2n-4)!}{3^2 2!1^{2n-7}(2n-7)!1^2 2!1^1 1!}\right)\\
&= \frac{1}{18}\,(2(2n+1)(n+1)n + 2(2n-1)(2n-2)(2n-3)\\
&\quad + 2(2n-4)(2n-5)(2n-6))\\
&= 2n^3 - 9n^2 + 19n - 14.
\end{aligned}$$

Now, let $n = 3$ or $n = 2$. Then

$$L\left(G_n, S_{2n-1} \times S_2 \times S_1\right) = \left\{\left(\left(3^1, 1^{2n-1}\right), \left(3^1, 1^{2n-4}\right), \left(1^2\right), \left(1^1\right)\right)\right\},$$

and $\nu_{(5,2,1);G_3} = 16$, $\nu_{(3,2,1);G_2} = 4$. The last two numbers are the values of the polynomial $2n^3 - 9n^2 + 19n - 14$ for $n = 3$, and $n = 2$, respectively.

(vi) Suppose that $n \geq 4$. We have

$$\begin{aligned}
L\left(G_n, S_{2n-1} \times S_1 \times S_1 \times S_1\right) = \Big\{&\left(\left(3^1, 1^{2n-1}\right), \left(3^1, 1^{2n-4}\right), \left(1^1\right), \left(1^1\right),\right.\\
&\left.\left(1^1\right)\right), \left(\left(3^2, 1^{2n-4}\right), \left(3^2, 1^{2n-7}\right), \left(1^1\right),\right.\\
&\left.\left(1^1\right), \left(1^1\right)\right)\Big\},
\end{aligned}$$

and hence we obtain

$$\begin{aligned}
\nu_{(2n-1,1^3);G_n} = \frac{1}{18}&\left(\frac{(2n+2)!}{(2n-1)!1!1!1!} + 4\frac{z_{(3^1,1^{2n-1})}}{z_{(3^1,1^{2n-4})}z_{(1^1)}z_{(1^1)}z_{(1^1)}}\right.\\
&\left. + 4\frac{z_{(3^2,1^{2n-4})}}{z_{(3^2,1^{2n-7})}z_{(1^1)}z_{(1^1)}z_{(1^1)}}\right)
\end{aligned}$$

$$= \frac{1}{18}\left((2n+2)(2n+1)(2n) + 4\frac{3^1 1! 1^{2n-1}(2n-1)!}{3^1 1! 1^{2n-4}(2n-4)!(1^1 1!)^3}\right.$$
$$\left. + 4\frac{3^2 2! 1^{2n-4}(2n-4)!}{3^2 2! 1^{2n-7}(2n-7)!(1^1 1!)^3}\right)$$
$$= \frac{1}{18}(2(2n+1)(n+1)n + 4(2n-1)(2n-2)(2n-3)$$
$$+ 4(2n-4)(2n-5)(2n-6)) = 4n^3 - 18n^2 + 38n - 28.$$

Now, let us suppose $n = 3$ or $n = 2$. Then

$$L(G_n, S_{2n-1} \times S_1 \times S_1 \times S_1) = \left\{\left(\left(3^1, 1^{2n-1}\right), \left(3^1, 1^{2n-4}\right), \left(1^1\right),\right.\right.$$
$$\left.\left.\left(1^1\right), \left(1^1\right)\right)\right\},$$

and $\nu_{(5,2,1);G_3} = 32$, $\nu_{(3,2,1);G_2} = 8$. The last two numbers are the values of the polynomial $4n^3 - 18n^2 + 38n - 28$ for $n = 3$, and $n = 2$, respectively.

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